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QUADRATIC AND λ-HERMITIAN FORMS

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Let X be a topological space and $C(X,k)$, $k=\mathbb{R}$, $\mathbb{C}$, $\mathbb{K}$, be the ring of continuous k-valued functions on X. We give algebraic conditions for a subring of $C(X,k)$ to have, up to isomorphism, the same quadratic or λ-hermitian forms.

Introduction

For a topological space X let $C(X,k)$, with $k=\mathbb{R}$, $\mathbb{C}$, $\mathbb{K}$ ($\mathbb{K}$ the field of the quaternions), be the ring of continuous k-valued functions on X, and $P(\ )$ the monoid of isomorphism classes of finitely generated projective ( )-modules where ( ) is any ring with unit.

In [3,7,8] some conditions are given for a subring R of $C(X,k)$ to have $P(R)\simeq P(C(X,k))$ and Swan links this problem with Hilbert's 17th problem. Here we show that these conditions link the λ-hermitian K_0 of R with that of $C(X,k)$; and thus with their Witt rings.

In [6] Evans proved that if A and B are two rings and $i:A\to B$ a morphism, then the canonical application from $P(A)$ to $P(B)$ defined by $P \rightsquigarrow P\otimes_A B$ is injective if the following conditions hold:

Evans's conditions. Let A and B be two rings with unit and $i:A\to B$ a ring homomorphism with $i(1)=1$.

(i) B is a topological ring, B^*, the group of units of B, is open in B, and $u \rightsquigarrow u^{-1}$ is continuous on B^*.

(ii) $i(A)$ is dense in B.

(iii) If $i(a)$ is sufficiently near 1 in B, then $a\in A^*$.

For the surjectivity Swan gives the extra condition (SBI_n) for all n:

Swan's condition (SBI_n). If U is a neighborhood of 0 in A, then there exists a neighborhood V of 0 in A with the following property:

If $z\in M_n(A)$, the ring of $n\times n$-matrices over A, has all entries in V, then there is some $x\in M_n(A)$ with all entries in U such that $x^2-x=z$.

In [3] the extra condition (SBI_n) for all n is substituted by: For all finitely generated projective B-modules Q there exist ideals $(\mathfrak{A}_i)_{i=1,\dots,n}$ of A such that

(a) $\bigcap_{i=1}^{n} \mathfrak{A}_i = 0$,

(b) $Q/\mathfrak{b}_i Q$ is a free $B/\mathfrak{b}_i$-module, for all $i=1,\dots,n$ where $\mathfrak{b}_i = \mathfrak{A}_i B$.

From now on the notations and the definitions will be taken from [1]; we just recall some of them:

By a ring with involution we mean a unitary ring with an antiautomorphism $a \rightsquigarrow \bar{a}$ of order ≤ 2 and we define over $M_n(A)$ the involution 'conjugate transpose' by $\overline{(a_i^j)} = (\overline{a_j^i})$ and for a sesquilinear form $h: N \times M \to A$ the sesquilinear form $\bar{h}: M \times N \to A$ by $\bar{h}(m,n) = \overline{h(n,m)}$.

Let λ be an element in the center of A such that $\lambda\bar{\lambda}=1$, and $h: M \times M \to A$ a sesquilinear form. We say that h is a λ-hermitian form if $h = \lambda\bar{h}$. A couple (M,q) will be a λ-hermitian A-module if M is a finitely generated projective A-module and q a non-singular λ-hermitian form. We denote $K_0^\lambda(A)$ the λ-hermitian K_0 of A (i.e. modulo the metabolic modules $(M,q) \perp (M,-q)$) and $W^\lambda(A)$ the induced Witt ring.

Condition P_λ. Let A be a ring with involution satisfying (i) of Evans's conditions and λ an element in the center of A such that $\lambda\bar{\lambda}=1$. Denote $A_\lambda = \{a \in A : a = \lambda\bar{a}\}$. We say that A satisfies the P_λ-condition if:

For U a neighborhood of 0 in A and a an element of A_λ, there exists a neighborhood V of 0 in A such that any equation of the form $\bar{T}aT + T + \lambda\bar{T} = \alpha$ has a solution in U if $\alpha \in V \cap A_\lambda$.

Condition (SBI_1) is the P_1-condition with the trivial involution when $\frac{1}{2} \in A$.

Examples. Any ring A with 2 invertible and the Evans's conditions satisfies P_{-1}.

Let X be a topological space and $C(X,k)$, $k=\mathbb{R}$ or $\mathbb{C}$, the ring of continuous k-valued functions on X. Then

(i) $C(X,\mathbb{R})$, with the trivial involution, satisfies P_1. Take the equation $aT^2 + 2T = \alpha$ with α sufficiently near 0. The element $1 + a\alpha$ is positive and

$$\beta = \frac{\alpha}{1 + \sqrt{1 + a\alpha}}$$

is a solution of the given equation.

(ii) $C(X,\mathbb{C})$, with the canonical involution 'conjugate', satisfies P_1. For $C(X,\mathbb{C})_1 = C(X,\mathbb{R})$ the proof is the same as before.

Other examples are given later.

1. Free modules

For a ring B satisfying the P_λ-condition take $M = (b_j^i)$, $i=1,\dots,n$, $j=1,\dots,n$, a matrix with all entries in B such that $M=\lambda\bar{M}$, $M^{-1}=(C_j^i)$ the inverse matrix of M, and ε an element of B.

Let δ_j^i be the element 1 if $i=j$ and 0 otherwise. We define the matrix $E_{j_0}^{i_0}(\varepsilon)=(s_j^i)$ by $s_j^i=\delta_j^i+\delta_{j_0}^j c_{i_0}^i \varepsilon$.

Lemma 1.1. *The matrix* $M'=\overline{E_{i_0}^{j_0}(\varepsilon)}ME_{j_0}^{i_0}(\varepsilon)=(b_j'^i)$ *is of the form*:

(i) *If* $i_0=j_0$, *then* $b_{j_0}'^{j_0}=b_{j_0}^{j_0}+\bar{\varepsilon}\overline{C_{j_0}^{j_0}}\varepsilon+\lambda\bar{\varepsilon}+\varepsilon$, *and* $b_j'^i=b_j^i$ *for all* $(i,j)\neq(j_0,j_0)$.

(ii) *If* $i_0\neq j_0$, *then* $b_{j_0}'^{i_0}=b_{j_0}^{i_0}+\varepsilon$, $b_{i_0}'^{j_0}=b_{i_0}^{j_0}+\lambda\bar{\varepsilon}$, $b_{j_0}'^{j_0}=b_{j_0}^{j_0}+\bar{\varepsilon}\overline{C_{j_0}^{j_0}}\varepsilon$, *and* $b_j'^i=b_j^i$ *otherwise.*

It is clear that $M'=\lambda\bar{M}'$ and if ε is sufficiently near 0, the matrix $E_{j_0}^{i_0}(\varepsilon)$ is invertible. Two matrices are equivalent if there exists an invertible matrix P such that $M'=\bar{P}MP$.

Proposition 1.2. *Let B be a ring with the* P_λ*-condition. All matrices M and M' such that $M=\lambda\bar{M}$ and $M'=\lambda\bar{M}'$ are sufficiently near, are equivalent.*

In other words the orbit under the action of $\mathrm{GL}(n,B)$ of a non-degenerated matrix M of $M_n(B)$ such that $M=\lambda\bar{M}$ is open.

Proof. Let $M=(b_j^i)$, $M^{-1}=(C_j^i)$, $M'=(a_j^i)$ and $\varepsilon_j^i=a_j^i-b_j^i$ for $i=1,\dots,n$, $j=1,\dots,n$.

Since the matrices M and M' are sufficiently near we show that there exists an invertible matrix P such that $\bar{P}MP=(d_j^i)$ with $d_j^i=a_j^i$ for $i=1$ or $j=1$ and with $D=(d_j^i)$, $i=2,\dots,n$, $j=2,\dots,n$, a matrix sufficiently near (a_j^i), $i=2,\dots,n$, $j=2,\dots,n$, satisfying $D=\lambda\bar{D}$.

Take α_1^1 a solution near 0 of the equation $\bar{X}\bar{C}_1^1X+X+\lambda\bar{X}=\varepsilon_1^1$. Then the matrix $P=E_1^1(\alpha_1^1)E_2^1(\varepsilon_2^1)\dots E_n^1(\varepsilon_n^1)$ agrees with what we want.

By induction, using the same process, the proposition is proved. □

By a subring we mean a subring stable by involution with the same 1.

Proposition 1.3. *Let B be a ring with the* P_λ*-condition and A a dense subring of B with λ in the center of A, A_λ dense in B_λ, and $A^*=B^*\cap A$. Then any invertible matrix M with entries in B such that $M=\lambda\bar{M}$ is equivalent to an invertible matrix M' with all entries in A. The matrix M' can be chosen arbitrarily near M.*

We use the density of A in B to approximate the elements of M and Proposition 1.2.

Corollary 1.4. *Let B be a ring with the* P_λ*-condition and A a dense subring of B with $\lambda\in A$ and $A^*=B^*\cap A$. If A, with the induced topology, satisfies the* P_λ*-condition, then all invertible matrices M and M' with entries in A such that $M=\lambda\bar{M}$, $M'=\lambda\bar{M}'$ equivalent in $M_n(B)$ are equivalent in $M_n(A)$.*

Proof. There exists an invertible matrix P in $M_n(B)$ such that $M' = \bar{P}MP$. We take an invertible matrix N, sufficiently near I_n such that PN has all entries in A as in [3]. Then $M'' = \bar{N}M'N = (\overline{PN})M(PN)$ is equivalent to M in $M_n(A)$; since M' and M'' are sufficiently near, they are equivalent in $M_n(A)$. $\square$

2. Projective modules

For a ring R satisfying the P_λ-condition we look for the possibility of having a non-singular λ-hermitian form on a finitely generated projective R-module.

Lemma 2.1. *Let R be a ring with $\frac{1}{2} \in R$ and q an element of $W^\lambda(R)$. Then $2q$ is represented by a free module.*

Lemma 2.2. *For every λ-hermitian R-module (P, b) there exists a λ-hermitian R-module (Q, g) such that $P \oplus Q$ is free.*

Proof. Let (P, g) be an element not in the class of q; b induces an isomorphism from P to $\bar{P}$ and $(\bar{P}, \lambda\bar{b})$ is within the class of q. Take Q a finitely generated projective R-module such that $P \oplus Q \simeq R^n$ and h is the hyperbolic form on $Q \oplus \bar{Q}$. Then $2q$ is represented by $(P, B) \perp (\bar{P}, \lambda\bar{b}) \perp (Q \oplus \bar{Q}, h)$.

Since the form $\lambda\bar{b} \oplus h$ is non-singular over $\bar{P} \oplus Q \oplus \bar{Q}$, the lemma follows. $\square$

Lemma 2.3. *Let A be a subring of a ring B, P a finitely generated projective A-module and b a non-singular λ-hermitian form over $P' = P \otimes_A B$. Then there exists a finitely generated projective A-module M such that $P \oplus M$ is free, and g is a non-singular λ-hermitian form over $M' = M \oplus_A B$.*

Proof. Let Q be such that $P \oplus Q \simeq A^n$, then

$$(P', b) \perp (\overline{P'}, \lambda\bar{b}) \perp (Q' \oplus \overline{Q'}, h) \simeq (P', b) \perp (\bar{P}', \lambda\bar{b}) \perp ((Q \oplus \bar{Q})', h)$$

where h is the hyperbolic form and $(\ )'$ means $(\ ) \otimes_A B$. Obviously $M = \bar{P} \oplus Q \oplus \bar{Q}$ and $g = \lambda\bar{b} \oplus h$ agree. $\square$

Lemma 2.4. *Let $\varphi : A \to B$ be a ring homomorphism with Evans's conditions and b a λ-hermitian form over a finitely generated projective A-module P such that $b' = b \otimes 1$. Then b is non-singular.*

For the notations we refer to [1].

Proof. *d_b (resp. b^d) is injective.* If $d_b(y) = 0$ we have $0 = j_p \circ (d_b(y) \otimes 1) = d_{b \otimes 1}(y \otimes 1)$, so $y \otimes 1 = 0$. To see that $y = 0$ we can take Q such that $P \oplus Q$ is free, and use a basis of it.

d_b *(resp.* b^d*) is surjective.* Let Im( ) be the image of ( ). Since b' is non-singular, $d_b \otimes 1$ is an isomorphism so $(\bar{P}/\operatorname{Im} d_b)\otimes_A B = 0$. By [8, Lemma 2.4] we have $\bar{P}/\operatorname{Im} d_b = 0$. □

Proposition 2.5. *Let B be a ring with the P_λ-condition and A a dense subring of B with λ in the center of A, A_λ dense in B_λ, and $A^* = B^* \cap A$. Suppose that, with the induced topology, the P_λ-condition holds in A. Let P be a finitely generated projective A-module and f a non-singular λ-hermitian form over $P' = P \otimes_A B$. Then there exists over P a non-singular λ-hermitian form b such that $b \otimes 1$ is arbitrarily near f.*

Proof. Take (Q', g) to be a λ-hermitian module such that $P \oplus Q \simeq A^n$. Then $f \oplus g$ is represented by an invertible λ-hermitian matrix M which is equivalent to a λ-hermitian matrix arbitrarily near M, with all entries in A. This matrix induces over P a λ-hermitian form b such that $b \otimes 1$ is arbitrarily near f. By [8, Lemma 1.3], $b \otimes 1$ is non-singular, and b is non-singular. □

3. The morphism $K_0^\lambda(\varphi)$

For B a ring with $\frac{1}{2} \in B$, satisfying the P_λ-condition, and A a dense subring of B with $\lambda \in A$ and $A^* = B^* \cap A$, it is easy to see that the cokernel of the canonical map $\psi : W^\lambda(A) \to W^\lambda(B)$ is 2-torsion. If moreover the P_λ-condition holds in A, with the induced topology, then the kernel of ψ is 2-torsion.

In the case that $K_0(\varphi)$, where $\varphi : A \to B$, is an isomorphism, we can be more precise for $K_0^\lambda(\varphi)$ and ψ (here we need not know that $\frac{1}{2} \in B$).

Definition 3.1. Two λ-hermitian R-modules (P, f), (Q, g) are *sufficiently near* if $P \simeq^\varphi Q$ and if d_f (resp. f^d) and d_g (resp. g^d) are sufficiently near 'modulo φ', i.e. we compare f and $g \circ (\varphi \times \varphi)$.

Lemma 3.2. *Let R be a ring satisfying the P_λ-condition and (P, f) and (Q, g) two λ-hermitian R-modules sufficiently near. Then their images in $K_0^\lambda(R)$ are equal.*

Proof. Let (H, h) be a λ-hermitian R-module such that $P \oplus H \simeq \mathbb{R}^n \simeq Q \oplus H$. The matrices of $(f \oplus h)$ and $(g \oplus h)$ are equivalent. □

Proposition 3.3. *Let B be a ring satisfying the P_λ-condition and A a subring of B with λ in the center of A dense in B_λ. Suppose that the canonical injection $\varphi : A \to B$ satisfies the Evans's conditions and $K_0(\varphi) : K_0(A) \to K_0(B)$ is bijective. Then $K_0^\lambda(\varphi) : K_0^\lambda(A) \to K_0^\lambda(B)$ and $\psi : W^\lambda(A) \to W^\lambda(B)$ are both surjective. Moreover if the P_λ-condition holds in A with the induced topology, then $K_0^\lambda(\varphi)$ and ψ are bijective.*

Proof. From Proposition 2.5 and Lemma 3.2 the surjectivity is immediate.

Let (P,f) and (Q,g) be two λ-hermitian A-modules such that their images are equal in $K_0^\lambda(B)$. There exists a λ-hermitian B-module (H',h') such that $(P,f)'\oplus(H',f')\simeq(Q,g)'\oplus(H',f')$. Evidently we can choose H' to be a free B-module of rank n and h' to be induced by a non-singular λ-hermitian A-form h.

Take a λ-hermitian A-module (R,r) such that $P\oplus R\simeq A^m$, then $Q\oplus R\simeq A^m$ and

$$(P,f)'\oplus(R,r)'\oplus(A^n,h)'\simeq(Q,g)'\oplus(R,r)'\oplus(A^n,h)'.$$

By Corollary 1.4 we have $(P,f)\oplus(R,r)\oplus(A^n,h)\simeq(Q,g)\oplus(R,r)\oplus(A^n,h)$. $\square$

Some examples for quadratic forms

Let X be a subspace of $\mathbb{R}^n$.

Example 1. By [8] the conditions of Proposition 3.3 hold for A the ring of Nash functions, and B the ring of semi-algebraic continuous functions where X is compact or open semi-algebraic; in fact they hold for X a subvariety of Nash of $\mathbb{R}^n$.

Example 2. If X is sufficiently generally embedded in $\mathbb{R}^n$ with the vertices in general position and linearly independent, then we can choose A to be the ring $\Gamma^1(X)$, or $\Gamma^\infty(X)$, of [2] and B the ring $C(X)$ ($\Gamma^1(X)$ is the ring generated by $\Gamma^0(X)$, the ring of the polynomial functions on X, and the elements $\sqrt{f}$, $1/f$ where $f\in\Gamma^0(X)$ and f is never nil on X).

Appendix

As for vector bundles over paracompact space we have:

Proposition. *Let A be a subring with Hausdorff maximal spectrum of $C(X,\mathbb{R})$. Then over any finitely generated projective A-module P there exists a non-singular λ-hermitian A-form g (with the trivial involution).*

Proof. In the following we consider the case $\lambda=\pm 1$. Let $(U_i)_{i=1,\ldots,n}$ be an open covering of Spec A such that, over each U_i, the restriction of P is trivial. We can find an open subcovering $(V_i)_{i=1,\ldots,n}$, stable by generation, and elements $(f_i)_{i=1,\ldots,n}$ of A such that $\sum_{i=1}^n f_i=1, f_i\geq 0$ for $i=1,\ldots,n$, and $\overline{D(f_i)}\subset V_i$. Obviously there exists a non-singular λ-hermitian form g_i over P/V_i for all i. We take $g=\sum_{r=1}^n f_i g_i$.

Note. We have a similar proposition for a subring of $C(X,\mathbb{C})$ with the involution 'conjugate'. For such a covering we can find functions $(f_i)_{i=1,\ldots,n}$ with real values such that $\sum f_i=1$, $f_i\geq 0$ and $\overline{D(f_i)}\subset V$ (take $f_i'=f_i\overline{f_i}/f$ where $f=\sum f_i\overline{f_i}$).

Examples. As a subring with Hausdorff maximal spectrum of $C(X, \mathbb{R})$ we have the ring of semialgebraic functions, or the ring of piecewise rational functions.

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